

# Supplementary Information for: Bosonic topological insulator intermediate state in the superconductor-insulator transition

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(Dated:)

## EFFECTIVE GAUGE THEORY FOR THE SIT IN FILMS

Let us consider elementary charges  $\bar{q}e$  and elementary vortices with flux  $\phi$  in (2+1) dimensions, so that both are point-like objects. The Aharonov-Bohm effect amounts to a quantum mechanical phase  $\bar{q}e\phi$  (throughout this Supporting Online Material we use natural units  $c = 1$ ,  $\hbar = 1$ ) acquired by the wave function when a charge encircles a vortex while the Aharonov-Casher effect, its dual, amounts to the same phase when a vortex encircles a charge.

In Euclidean field theory, the world-lines of point-like objects in (2+1) dimensions become curves in 3-dimensional space. The trajectory of a charge (vortex) encircling a vortex (charge) becomes simply a closed curve around a straight line along the original “time” direction. This configuration has a non-trivial Gauss linking number: you cannot “extract” the infinitely long straight line from the closed circle without breaking it. When summing over trajectories in the Euclidean path integral, the Aharonov-Bohm-Casher phases appear in the partition function as topological weights for the trajectory configurations measuring the Gauss linking number [1]. In the above example, this contribution is a factor  $\exp(i\bar{q}e\phi)$  in the Euclidean partition function. For general ensembles  $\{Q_\mu\}$  and  $\{M_\mu\}$  of trajectories of charges and vortices parametrized by  $q^{(i)}$  and  $m^{(i)}$ ,

$$\begin{aligned} Q_\mu &= \sum_i \int_{Q_i} d\tau \frac{dq_\mu^{(i)}(\tau)}{d\tau} \delta^3(x - q^{(i)}(\tau)) , \\ M_\mu &= \sum_i \int_{M_i} d\tau \frac{dm_\mu^{(i)}(\tau)}{d\tau} \delta^3(x - m^{(i)}(\tau)) , \end{aligned} \quad (1)$$

the contribution to the action in the partition function representing the topological ABC interactions becomes

$$S_{\text{linking}} = i\bar{q}e\phi \int d^3x Q_\mu \epsilon_{\mu\alpha\nu} \frac{\partial_\alpha}{-\nabla^2} M_\nu . \quad (2)$$

For closed curves this is the sum of the Gaussian linking numbers between the loops of the two kinds. When charges and vortices satisfy the Dirac quantization condition  $\bar{q}e\phi = 2\pi$ , as we will henceforth assume, the linking action reduces to

$$S_{\text{linking}} = 2\pi i \int d^3x Q_\mu \epsilon_{\mu\alpha\nu} \frac{\partial_\alpha}{-\nabla^2} M_\nu . \quad (3)$$

Because of the factor  $(2\pi i)$ , integer linking numbers between closed loops do not contribute to the partition function. They do, however for generic, infinitely extended world-lines of charges and vortices.

The linking invariant (3) of trajectories is non-local and cannot be used, as such, in a derivative expansion of an effective field theory. It was, however shown by [2] that this simplest of all knot invariants can be indeed formulated locally in terms of two emergent U(1) gauge fields  $a_\mu$  and  $b_\mu$  with a mixed Chern-Simons (CS) [3] interaction and Euclidean action given by

$$S = \int d^3x \, i \frac{\bar{q}}{2\pi} a_\mu \epsilon_{\mu\alpha\nu} \partial_\alpha b_\nu + i\sqrt{\bar{q}} a_\mu Q_\mu + i\sqrt{\bar{q}} b_\mu M_\mu . \quad (4)$$

To reproduce this result one introduces in the action, as gauge invariant regulators, two standard Maxwell terms for the two gauge fields,

$$S_{\text{regulated}} = \int d^3x \, \frac{1}{4e_a^2} f_{\mu\nu}^a f_{\mu\nu}^a + \frac{1}{4e_b^2} f_{\mu\nu}^b f_{\mu\nu}^b + i \frac{\bar{q}}{2\pi} a_\mu \epsilon_{\mu\alpha\nu} \partial_\alpha b_\nu + i\sqrt{\bar{q}} a_\mu Q_\mu + i\sqrt{\bar{q}} b_\mu M_\mu , \quad (5)$$

where  $f_{\mu\nu}^a = \partial_\mu a_\nu - \partial_\nu a_\mu$  and  $f_{\mu\nu}^b = \partial_\mu b_\nu - \partial_\nu b_\mu$  and  $e_a$  and  $e_b$  are the two corresponding coupling constants. One can now perform a Gaussian integration over the two gauge fields  $a_\mu$  and  $b_\mu$  to obtain the effective action for the trajectories  $Q_\mu$  and  $M_\mu$  alone. In the limit  $e_a \rightarrow \infty$  and  $e_b \rightarrow \infty$  in which the regulators are removed this gives exactly (3),

$$\begin{aligned} e^{-S_{\text{linking}}(Q_\mu, M_\mu)} &= \frac{1}{Z} \lim_{e_a \rightarrow \infty, e_b \rightarrow \infty} \int \mathcal{D}a_\mu \mathcal{D}b_\mu e^{-S_{\text{regulated}}(a_\mu, b_\mu, Q_\mu, M_\mu)} , \\ Z &= \lim_{e_a \rightarrow \infty, e_b \rightarrow \infty} \int \mathcal{D}a_\mu \mathcal{D}b_\mu e^{-S_{\text{regulated}}(a_\mu, b_\mu)} \end{aligned} \quad (6)$$

The mixed Chern-Simons term is thus the local formulation of the mutual phases acquired by charges (vortices) moving in the presence of vortices (charges). As has been pointed out in [3] these mutual phases are the *dominant* gauge interactions at large distances in (2+1) dimensions, since the CS term is the only marginal gauge invariant term possible in (2+1) dimensions.

The full effective action for the SIT must respect the two gauge invariances. The Chern-Simons term is the only marginal gauge invariant term in 2D since it is the unique gauge invariant term involving only one field derivative. Topological interactions thus dominate near the SIT. From a purely field-theoretic point of view, the charge and vortex world-lines represent the singularities in the dual field strengths  $f_\mu = \epsilon_{\mu\alpha\nu} \partial_\alpha b_\nu$  and  $g_\mu = \epsilon_{\mu\alpha\nu} \partial_\alpha a_\nu$  arising from the compactness of the two U(1) gauge groups [1]. The conserved currents  $j_\mu = (\sqrt{\bar{q}}/2\pi) f_\mu$  and  $\phi_\mu = (\sqrt{\bar{q}}/2\pi) g_\mu$  represent thus the number current fluctuations of charges and vortices, respectively.

The next order terms in the derivative expansion of the effective action (4) contain two derivatives: gauge invariance requires then that they must be built from the “electric” ( $f_i$  and  $g_i$ ) and “magnetic” ( $f_0$  and  $g_0$ ) fields of the two gauge potentials. The most general possible gauge invariant action up to two field derivatives is then given by

$$S = \int dt d^2x \, i \frac{\bar{q}}{2\pi} a_\mu \epsilon_{\mu\alpha\nu} \partial_\alpha b_\nu + \frac{v_c}{2e_v^2} f_0 f_0 + \frac{1}{2e_v^2 v_c} f_i f_i + \frac{v_c}{2e_q^2} g_0 g_0 + \frac{1}{2e_q^2 v_c} g_i g_i + i\sqrt{\bar{q}} a_\mu Q_\mu + i\sqrt{\bar{q}} b_\mu M_\mu , \quad (7)$$

with the magnetic permeability  $\mu_P$  and the electric permittivity  $\epsilon_P$ , which determine the speed of light  $v_c = 1/\sqrt{\mu_P \epsilon_P}$  in the material. The two coupling constants  $e_q^2$  and  $e_v^2$  are phenomenological parameters having the dimensionality of [1/length]. Upon addition of these kinetic terms, both gauge fields acquire a topological Chern-Simons mass  $m_T = \bar{q} e_q e_v / 2\pi v_c$ , entering the dispersion relation  $E = \sqrt{m_T^2 v_c^4 + v^2 p^2}$  (see Section 2 below).

## LATTICE CHERN-SIMONS OPERATOR

A proper formulation of a compact U(1) gauge theory requires the introduction of an ultraviolet lattice regularization [1]. Following [4] we introduce first the forward and backward derivatives and shift operators on a three-dimensional Euclidean lattice with sites denoted by  $\{x\}$ , directions indicated by Greek letters and lattice spacing  $\ell$ ,

$$\begin{aligned} d_\mu f(x) &= \frac{f(x + \ell \hat{\mu}) - f(x)}{\ell} , & S_\mu f(x) &= f(x + \ell \hat{\mu}) , \\ \hat{d}_\mu f(x) &= \frac{f(x) - f(x + \ell \hat{\mu})}{\ell} , & \hat{S}_\mu f(x) &= f(x - \ell \hat{\mu}) . \end{aligned} \quad (8)$$

Summation by parts on the lattice interchanges both the two derivatives (with a minus sign) and the two shift operators. Gauge transformations are defined by using the forward lattice derivative. In terms of these operators one can then define two lattice Chern-Simons terms

$$k_{\mu\nu} = S_\mu \epsilon_{\mu\alpha\nu} d_\alpha , \quad \hat{k}_{\mu\nu} = \epsilon_{\mu\alpha\nu} \hat{d}_\alpha \hat{S}_\nu , \quad (9)$$

where no summation is implied over equal indices. Summation by parts on the lattice interchanges also these two operators (without any minus sign). Gauge invariance is then guaranteed by the relations

$$k_{\mu\alpha}d_\nu = \hat{d}_\mu k_{\alpha\nu} = 0, \quad \hat{k}_{\mu\nu}d_\nu = \hat{d}_\mu \hat{k}_{\mu\nu} = 0. \quad (10)$$

Note that the product of the two Chern-Simons terms gives the lattice Maxwell operator

$$k_{\mu\alpha}\hat{k}_{\alpha\nu} = \hat{k}_{\mu\alpha}k_{\alpha\nu} = -\delta_{\mu\nu}\nabla^2 + d_\mu\hat{d}_\nu, \quad (11)$$

where  $\nabla^2 = \hat{d}_\mu d_\mu$  is the 3D Laplace operator. The discrete version of the mixed Chern-Simons gauge theory can thus be formulated as

$$\begin{aligned} S = \sum_x & i\frac{\ell_0\ell^2\bar{q}}{2\pi}a_\mu k_{\mu\nu}b_\nu + \frac{\ell_0\ell^2v_c}{2e_v^2}f_0^2 + \frac{\ell_0\ell^2}{2e_v^2v_c}f_i^2 + \frac{\ell_0\ell^2v_c}{2e_q^2}g_0^2 + \frac{\ell_0\ell^2}{2e_q^2v_c}g_i^2 \\ & + i\ell_0\sqrt{\bar{q}}a_0Q_0 + i\ell\sqrt{\bar{q}}a_iQ_i + i\ell_0\sqrt{\bar{q}}b_0M_0 + i\ell\sqrt{\bar{q}}b_iM_i, \end{aligned} \quad (12)$$

where the discrete dual field strengths are given by

$$f_\mu = k_{\mu\nu}b_\nu, \quad g_\mu = k_{\mu\nu}a_\nu, \quad (13)$$

and  $\ell_0 = \ell/v_c$ .

## QUANTUM PHASE STRUCTURE

This action (12) can be rewritten as

$$\begin{aligned} S = \sum_x & \frac{\ell^3}{2e_v^2} \left[ b_i \left( -\frac{1}{v^2}d_0\hat{d}_0 - \nabla_2^2 \right) \delta_{ij}b_j + b_id_i\hat{d}_jb_j \right] \\ & + \frac{\ell^3}{2e_v^2} \frac{1}{v_c^2} \left[ b_0 \left( -\frac{1}{v^2}d_0\hat{d}_0 - \nabla_2^2 + \frac{1}{v^2}d_0\hat{d}_0 \right) b_0 + b_0d_0\hat{d}_ib_i + b_id_i\hat{d}_0b_0 \right] \\ & + i\frac{\ell^3\bar{q}}{2\pi v_c}a_\mu k_{\mu\nu}b_\nu \\ & + \frac{\ell^3}{2e_q^2} \left[ a_i \left( -\frac{1}{v^2}d_0\hat{d}_0 - \nabla_2^2 \right) \delta_{ij}a_j + a_id_i\hat{d}_ja_j \right] \\ & + \frac{\ell^3}{2e_q^2} \frac{1}{v_c^2} \left[ a_0 \left( -\frac{1}{v^2}d_0\hat{d}_0 - \nabla_2^2 + \frac{1}{v^2}d_0\hat{d}_0 \right) a_0 + a_0d_0\hat{d}_ia_i + a_id_i\hat{d}_0a_0 \right] \\ & + i\ell_0\sqrt{\bar{q}}a_0Q_0 + i\ell\sqrt{\bar{q}}a_iQ_i + i\ell_0\sqrt{\bar{q}}b_0M_0 + i\ell\sqrt{\bar{q}}b_iM_i, \end{aligned} \quad (14)$$

where  $\nabla_2^2$  denotes the Laplacian over “spatial” indices, indicated by Latin letters (repeated latin letters mean, correspondingly, summation over “spatial indices” only). By introducing auxiliary rescaled fields

$$\begin{aligned} \tilde{a}_0 &= \frac{1}{v_c} a_0, & \tilde{a}_i &= a_i, \\ \tilde{b}_0 &= \frac{1}{v_c} b_0, & \tilde{b}_i &= b_i, \end{aligned} \quad (15)$$

a rescaled time derivative  $\tilde{d}_0 = (1/v_c)d_0$ , a correspondingly modified Chern-Simons operator  $\tilde{k}_{\mu\nu}$  containing this time derivative, the action (14) can be reformulated exactly as the relativistic action (5) above, but now expressed entirely in terms of the rescaled quantities. We can thus use exactly the same Gaussian integration (6) to obtain the effective action for the strings  $Q_\mu$  and  $M_\mu$ . The real part of this action, the one that enters the determination of the phase structure, is given by

$$S_{\text{top}}^{\text{real}} = \sum_x v_c^2 \frac{e_q^2}{\ell} Q_\mu \frac{1}{v_c^4 m_T^2 - d_0\hat{d}_0 - v_c^2 \nabla_2^2} Q_\mu + v_c^2 \frac{e_v^2}{\ell} M_\mu \frac{1}{v_c^4 m_T^2 - d_0\hat{d}_0 - v_c^2 \nabla_2^2} M_\mu, \quad (16)$$

where we have set  $\bar{q} = 2$  for the relevant case of Cooper pairs.

In order to proceed we follow the standard arguments of [5] to retain only the self-interaction terms in (16). Consider closed strings made of  $N$  bonds, with integer quantum numbers  $Q_\mu = Q$  and  $M_\mu = M$  on all the lattice bonds forming the string and zero elsewhere. Such configurations can be assigned an energy (equivalent to Euclidean action in statistical field theory)

$$S_{\text{top}} = \pi m_T \ell v_c G(m_T \ell v_c) \left[ \frac{e_q}{e_v} Q^2 + \frac{e_v}{e_q} M^2 \right] N, \quad (17)$$

where  $G(m_T \ell v_c)$  is the diagonal element of the lattice kernel  $G(m_T \ell v_c, x - y)$  representing the inverse of the operator  $(\ell^2/v_c^2)(m_T^2 v_c^4 - d_0 \hat{d}_0 - v_c^2 \nabla_2^2)$ . The kernel  $G(m_T \ell v_c, x)$  is defined by the equation

$$(\ell^2/v_c^2)(m_T^2 v_c^4 - d_0 \hat{d}_0 - v_c^2 \nabla_2^2) G(m_T \ell v_c, x) = \delta_{x,0}. \quad (18)$$

Using the Fourier transform  $G(m_T \ell v_c, x) = \frac{\ell^3}{v_c} \int_{-\pi v_c/\ell}^{+\pi v_c/\ell} dk_0 \int_{-\pi/\ell}^{+\pi/\ell} d^2 k G(m_T \ell v_c, k) \exp(ik \cdot x)$  we obtain

$$\frac{\ell^3}{v_c} \int_{-\pi v_c/\ell}^{+\pi v_c/\ell} dk_0 \int_{-\pi/\ell}^{+\pi/\ell} d^2 k G(m_T \ell v_c, k) \frac{\ell^2}{v_c^2} (m_T^2 v_c^4 - d_0 \hat{d}_0 - v_c^2 \nabla_2^2) e^{ik \cdot x} = \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} d^3 k e^{ik \cdot x}. \quad (19)$$

Applying finally the finite difference operator  $(\ell^2/v_c^2)(m_T^2 v_c^4 - d_0 \hat{d}_0 - v_c^2 \nabla_2^2)$  to the exponential in the Fourier transform and rescaling momenta gives the final result

$$G(m_T \ell v_c) = \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} d^3 k \frac{1}{(m_T \ell v_c)^2 + \sum_{i=0}^2 4 \sin\left(\frac{k_i}{2}\right)^2}. \quad (20)$$

The string entropy, however is also proportional to their length, being given by  $\mu N$  with  $\mu \approx \ln(5)$  since at each step the non-backtracking strings can choose among 5 possible directions on how to continue. One can thus assign the free energy

$$F = \pi m_T \ell v_c G(m_T \ell v_c) \left[ \frac{e_q}{e_v} Q^2 + \frac{e_v}{e_q} M^2 - \frac{1}{\eta} \right] N, \quad (21)$$

to a string of length  $L = \ell N$  carrying electric and magnetic quantum numbers  $Q$  and  $M$ , respectively. Here we have introduced the dimensionless parameter

$$\eta = \frac{\pi m_T \ell v_c G(m_T \ell v_c)}{\mu}, \quad (22)$$

which, together with the ratio  $g = e_v/e_q$  fully determines the quantum phase structure, as we now show.

The ground state of the quantum model is found by minimizing its free energy as a function of  $N$ . When the energy term in (21) dominates, the free energy is positive and consequently minimized by short closed loop configurations. When, instead, the entropy dominates, the free energy is negative and minimized by large strings and long closed loops. The condition for condensation of long strings with integer quantum numbers  $Q$  and  $M$  is thus given by

$$\eta \frac{e_q}{e_v} Q^2 + \eta \frac{e_v}{e_q} M^2 < 1. \quad (23)$$

If two or more condensations are allowed, one has to choose the one with the lowest free energy. This condition describes the interior of an ellipse with semiaxes

$$\begin{aligned} r_Q &= \sqrt{\frac{e_v}{e_q} \frac{1}{\eta}}, \\ r_M &= \sqrt{\frac{e_q}{e_v} \frac{1}{\eta}}, \end{aligned} \quad (24)$$

on a square lattice of integer electric and magnetic charges. The phase diagram is consequently found by simply recording which integer charges lie within the ellipse when the semi-axes are varied,

$$\begin{aligned} \eta < 1 &\rightarrow \begin{cases} g > 1, \text{electric condensation} = \text{superconductor}, \\ g < 1, \text{magnetic condensation} = \text{superinsulator}, \end{cases} \\ \eta > 1 &\rightarrow \begin{cases} g > \eta, \text{electric condensation} = \text{superconductor}, \\ \eta > g > \frac{1}{\eta}, \text{no condensation} = \text{Bose metal}, \\ g < \frac{1}{\eta}, \text{magnetic condensation} = \text{superinsulator}, \end{cases} \end{aligned}$$

The new parameter  $\eta$  is responsible for the possible opening of a Bose metal phase between superconductor and superinsulator. In the vicinity of the SIT, where  $e_q \simeq e_v$ , it can be expressed as

$$\eta = \frac{\pi}{\mu} \frac{\pi}{\alpha} \frac{\ell}{\lambda_{\perp}} G \left( \frac{\pi}{\alpha} \frac{\ell}{\lambda_{\perp}} \right) , \quad (25)$$

where  $\alpha = e^2/\hbar c$  is the fine structure constant.

### TRANSITIONS INDUCED BY AN EXTERNAL MAGNETIC FIELD

In concrete experimental settings the phase transitions in thin films are driven either by varying the thickness of the films (disorder-driven transitions) or by varying an applied magnetic field. Varying the film thickness amounts effectively to varying the parameter  $g$  of the corresponding array. In the latter situation, instead there is one additional parity (P) and time-reversal (T) breaking external parameter that must be accounted for in the gauge theory formulation.

A uniform external magnetic field can be simply incorporated by the minimal coupling  $i\ell^3(2e\sqrt{2}/2\pi)A_{\mu}f_{\mu}$  to the charge current  $(\sqrt{2}/2\pi)f_{\mu}$ . By a summation by parts this amounts to the following additional term in the gauge theory action (7),

$$S \rightarrow S + \sum_x i\ell\sqrt{2} b_0 f , \quad (26)$$

where  $f$  denotes the number of elementary fluxes  $\pi/e$  per plaquette piercing the array. This modification amounts simply to shifting the integers

$$M_{\mu} \rightarrow M_{\mu} + \Phi_{\mu} , \quad (27)$$

in the original gauge theory model, where  $\Phi_{\mu}$  represents infinitely long strings in the Euclidean time direction at each lattice point, such that  $\Phi_i = 0$  and  $\Phi_0 = f$ . This shows that  $f$  is a periodic parameter defined modulo an integer: it is normally called the magnetic frustration,  $0 \leq f < 1$ , which explains why it is usually denoted by  $f$ .

An external magnetic field corresponds thus to a special case of condensed magnetic strings with non-integer quantum number. To incorporate the additional frustration parameter  $f$  we modify thus the string free energy to

$$F = \pi m_T \ell v_c^2 G(m_T \ell v_c) \left[ \frac{e_q}{e_v} Q^2 + \frac{e_v}{e_q} (M + f)^2 - \frac{1}{\eta} \right] N , \quad (28)$$

and to compare with the experimentally relevant situation let us assume we start with  $f = 0$  in the superconducting phase with condensation of electric strings,

$$\begin{aligned} \eta < 1 & \rightarrow g > 1 , \\ \eta > 1 & \rightarrow g > \eta . \end{aligned} \quad (29)$$

In this case the original  $f = 0$  ellipse is elongated along the electric quantum number axis. Turning on an external magnetic field amounts to increasing the frustration parameter  $f$  and, as consequence the ellipse moves down along the magnetic quantum number axis.

There are two important thresholds in this downward movement. The points on the ellipse corresponding to  $Q = \pm 1$  have  $M$  coordinate  $M = r_M \sqrt{1 - (1/r_Q^2)}$ , where  $r_Q$  and  $r_M$  are the semi-axes. The  $M$ -negative ellipse point with  $Q = 0$ , instead lies at a distance  $r_M$  from the origin by definition. A transition can be caused either by the fact that the two points  $Q = \pm 1$  will “exit” the interior of the ellipse before the point  $M = -1$  has “entered” it, in which case the initial superconductor turns into a bosonic topological insulator/Bose metal, or by the fact that the point  $M = -1$  has “entered” the ellipse interior while the two points  $Q = \pm 1$  are still inside. In this case we have a coexistence regime of electric and magnetic strings. There will thus be a first-order direct transition to a superinsulator when the frustration reaches a point such that the energy of the magnetic strings becomes smaller than that of the electric ones.

Clearly the two thresholds are given by the frustrations

$$\begin{aligned} f_1 &= r_M \sqrt{1 - \frac{1}{r_Q^2}} , \\ f_2 &= 1 - r_M . \end{aligned} \quad (30)$$

and the condition for an intermediate bosonic topological insulator/Bose metal phase is  $f_1 < f_2$ . Using the explicit expressions for the semiaxes in terms of array parameters we can translate this conditions into

$$F(g) \equiv \frac{1}{g^2} - \frac{2}{\sqrt{g\eta}} + 1 > 0 . \quad (31)$$

Let us first consider the case  $\eta < 1$  and let us express

$$\frac{1}{g} = 1 - \epsilon , \quad (32)$$

to satisfy (29). In this case the function  $F$  reduces to

$$F(g) = 2 \left( 1 - \frac{1}{\sqrt{\eta}} \right) - O(\epsilon) < 0 , \quad (33)$$

which shows that, in this case there is a direct transition from a superconductor to a superinsulator. For small  $f$ , the transition takes place when the new effective magnetic semiaxis  $r_M = r_M(1 + f)$  of the ellipse becomes larger than the originally larger electric semiaxis  $r_Q$ . This gives the critical frustration

$$f_{\text{crit}} = \frac{1}{2} (g^2 - 1) . \quad (34)$$

Let us now consider the second case  $\eta > 1$ . In this case (29) requires

$$\frac{1}{g} = \frac{1}{\eta} - \epsilon , \quad (35)$$

and the function  $F$  becomes

$$F(g) = \left( \frac{1}{\eta} - 1 \right)^2 - O(\epsilon) > 0 , \quad (36)$$

confirming that in this case, instead the transition from the superconductor is to a bosonic topological insulator/Bose metal phase. In this case the critical frustration is determined by the value at which the original electric topological excitations exit the shifted ellipse,

$$f_{\text{crit}} = f_1 = \frac{1}{\sqrt{g}} \sqrt{\frac{1}{\eta} - \frac{1}{g}} . \quad (37)$$

We have derived that an external magnetic field can induce the transition from a superconductor to a bosonic topological insulator. Let us now analyze what happens to this bosonic topological insulator if the magnetic field is further increased. In the bosonic topological insulator phase, the topological excitations are absent: as a consequence we can diagonalize the doubled Chern-Simons model (7) by the transformation

$$\begin{aligned} \tilde{a} &= \frac{1}{\sqrt{2}} \left( \sqrt{\frac{e_q}{e_v}} b - \sqrt{\frac{e_v}{e_q}} a \right) , \\ \tilde{b} &= \frac{1}{\sqrt{2}} \left( \sqrt{\frac{e_q}{e_v}} b + \sqrt{\frac{e_v}{e_q}} a \right) . \end{aligned} \quad (38)$$

where we have adopted, for simplicity, the continuum notation. In these new variables the action decouples into two free modes of mass  $m = \bar{q}e_v e_q / 2\pi$ ,

$$\begin{aligned} S &= \int d^3x \frac{1}{4e_v e_q} \tilde{f}_{\mu\nu} \tilde{f}_{\mu\nu} + i \frac{\bar{q}}{4\pi} \tilde{b}_\mu \epsilon^{\mu\nu\alpha} \partial_\nu \tilde{b}_\alpha \\ &+ \int d^3x \frac{1}{4e_v e_q} \tilde{g}_{\mu\nu} \tilde{g}_{\mu\nu} - i \frac{\bar{q}}{4\pi} \tilde{a}_\mu \epsilon^{\mu\nu\alpha} \partial_\nu \tilde{a}_\alpha . \end{aligned} \quad (39)$$

These modes describe fluctuations of composites of charge  $\bar{q}$  and fluxes  $\pm 2\pi/\bar{q}$ , the pure Chern-Simons term describing the Aharonov-Bohm phase  $2\pi$  picked up when one of these composites encircles another, as described in detail above. When an external magnetic field is introduced, however, a charge looping around an elementary plaquette picks up

an additional phase  $\pm 2\pi f$  depending on the sign of the frustration. In other words, the “effective flux” of one of the composites is increased while the other is decreased, an effect captured to first order by two effective  $\bar{q}_\pm = \bar{q} \pm f$ . An external magnetic field leads thus to a mass splitting

$$m \rightarrow m_\pm = \frac{e_v e_q}{\pi} \left( 1 \pm \frac{f}{2} \right). \quad (40)$$

In particular, the mass gap characterizing the bosonic topological insulator is lowered by an external magnetic field. When this effect is sufficiently strong we expect thus a “defrosting transition” leading to normal metallic behaviour.

### CONDUCTION BY EDGE MODES

The Chern-Simons effective action is not invariant under gauge transformations  $a_i = \partial_i \lambda$  and  $b_i = \partial_i \chi$  at the edges. Two chiral bosons [6]  $\lambda = \xi + \eta$  and  $\chi = \xi - \eta$  have to be introduced to restore the full gauge invariance, exactly as it is done in the quantum Hall effect framework [7] and for topological insulators [8]. The full gauge invariance is restored by adding the edge action

$$S_{\text{edge}} = \frac{1}{\pi} \int d^2x \left( \partial_0 \xi \partial_s \xi - \partial_0 \eta \partial_s \eta \right) + \bar{q} e_{\text{eff}} \int d^2x A_0 \left( \frac{\sqrt{\bar{q}}}{2\pi} \partial_s \chi \right), \quad (41)$$

including the electromagnetic coupling of the edge charge density  $\rho = (\bar{q} e_{\text{eff}})(\sqrt{\bar{q}}/2\pi) \partial_s \chi$  in the  $A_s = 0$  gauge,  $\bar{q} e_{\text{eff}}$  being the effective charge of the Cooper pairs in the Bose metal phase,  $\bar{q} e_{\text{eff}} = \bar{q} e \sqrt{g}$ . As in the case of the quantum Hall effect, the non-universal dynamics of the edge modes is generated by boundary effects [7], which result in the Hamiltonian

$$H = \frac{1}{\pi} \int ds \left[ -v_b (\partial_s \xi)^2 - v_b (\partial_s \eta)^2 \right], \quad (42)$$

where  $v_b$  is the velocity of propagation of the edge modes along the boundary. Upon adding this term, the total edge action becomes

$$S_{\text{edge}} = \frac{1}{\pi} \int d^2x \left[ (\partial_0 - v_b \partial_s) \xi \partial_s \xi - (\partial_0 + v_b \partial_s) \eta \partial_s \eta \right] + \bar{q} e_{\text{eff}} \int d^2x A_0 \left( \frac{\sqrt{\bar{q}}}{2\pi} \partial_s \chi \right). \quad (43)$$

The equation of motion generated by this action is

$$v_b \partial_s \rho = \frac{\bar{q} e_{\text{eff}}}{2\pi} E = \frac{\bar{q} e_{\text{eff}}}{2\pi} \partial_s A_0, \quad (44)$$

which represent ballistic charge conduction with the resistance  $R = R_Q/g$  with  $R_Q = h/(\bar{q}e)^2$  the quantum resistance.

### BOSONIC TOPOLOGICAL INSULATORS AS FUNCTIONAL FIRST LANDAU LEVELS

Let us consider the 2D mixed Chern-Simons theory (in Minkowski space in the following)

$$\mathcal{L} = \frac{\bar{q}}{2\pi} a_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha b_\nu, \quad (45)$$

which appears naively as the relevant effective action for the intermediate bosonic topological insulator phase [8]. First of all we would like to point out that, when the two gauge symmetries are considered as compact  $U(1)$ , this action cannot be diagonalized into two separate single CS terms. There are various ways to see this. First, a proper formulation of compact Abelian gauge theories requires either spontaneous symmetry breaking from a larger compact group or the formulation on a lattice [1]. In both cases, diagonalization is impossible. Otherwise it suffices to realize that the dimension of the “Brillouin zone” for one of the two diagonalizing gauge field combinations would depend on the other.

To quantize the mixed CS theory in the functional Schrödinger picture, one must pay careful attention to an anomaly affecting Chern-Simons theories [9], namely the gauge invariant states of this purely topological field theory differ

from the  $m \rightarrow \infty$  limit of the corresponding topologically massive gauge theory [3] due to non-commutativity between quantization and phase space reduction. Only the latter make physical sense since the former are not normalizable. This means that CS theories must always be considered as the  $m \rightarrow \infty$  limit of a topologically massive gauge theory, which necessarily introduces an often neglected additional dimensionless parameter.

We shall consider the following regularized field theory model,

$$\mathcal{L} = -\frac{1}{4e_v^2} f_{0i} f^{0i} + \frac{\bar{q}}{2\pi} a_\mu \epsilon^{\mu\alpha\nu} \partial_\alpha b_\nu - \frac{1}{4e_q^2} g_{0i} g^{0i} . \quad (46)$$

where  $f_{\mu\nu} = (\partial_\mu b_\nu - \partial_\nu b_\mu)$  and  $g_{\mu\nu} = (\partial_\mu a_\nu - \partial_\nu a_\mu)$ . This model corresponds to the deep non-relativistic limit of the general gauge theory model (7), in which the electric permittivity becomes so large that only "electric" fields  $f_{0i}$  and  $g_{0i}$  matter (as usual we use greek letters for space-time indices and latin letters for space indices) and magnetic effects can be neglected, which is exactly the relevant limit near the SIT [10, 11]. In this limit, the magnetic terms can be omitted and the electric permittivity  $\epsilon_P$  can be incorporated in the definition of the phenomenological parameters  $e_q$  and  $e_v$ . Topological insulators are recovered in the limit of infinite topological mass  $m \rightarrow \infty$ , where  $m = \bar{q}e_q e_v / 2\pi$ .

As usual for a gauge theory, the gauge components  $a^0$  and  $b^0$  are not dynamical fields, since they never appear with time derivatives. They are Lagrange multipliers, whose associated Gauss law constraints implement gauge invariance. They can be set to zero,  $a^0 = 0$  and  $b^0 = 0$ , after imposing the corresponding Gauss law constraints. This is called the Weyl gauge. The two canonical momenta conjugate to the canonical variables  $a^i$  and  $b^i$  are:

$$\begin{aligned} \mathcal{P}_a^i &= \frac{\delta \mathcal{L}}{\delta(\partial_0 a^i)} = \frac{1}{e_q^2} g^{0i} + \frac{\bar{q}}{4\pi} \epsilon^{ij} b^j , \\ \mathcal{P}_b^i &= \frac{\delta \mathcal{L}}{\delta(\partial_0 b^i)} = \frac{1}{e_v^2} f^{0i} + \frac{\bar{q}}{4\pi} \epsilon^{ij} a^j . \end{aligned} \quad (47)$$

They are realized as functional derivatives,

$$\mathcal{P}_a^i = -i \frac{\delta}{\delta a^i} , \quad \mathcal{P}_b^i = -i \frac{\delta}{\delta b^i} . \quad (48)$$

The Hamiltonian density, when written in canonical variables takes the form

$$\mathcal{H} = \frac{e_q^2}{2} (\Pi_a^i)^2 + \frac{e_v^2}{2} (\Pi_b^i)^2 , \quad (49)$$

where

$$\begin{aligned} \Pi_a^i &= \mathcal{P}_a^i - \frac{\bar{q}}{4\pi} \epsilon^{ij} b^j , \\ \Pi_b^i &= \mathcal{P}_b^i - \frac{\bar{q}}{4\pi} \epsilon^{ij} a^j , \end{aligned} \quad (50)$$

are the kinetic momenta. Due to the Chern-Simons term, the kinetic momenta do not commute,

$$[\Pi_a^i(\mathbf{x}), \Pi_b^j(\mathbf{y})] = -i \frac{\bar{q}}{2\pi} \epsilon^{ij} \delta^2(\mathbf{x} - \mathbf{y}) . \quad (51)$$

This shows, that the Chern-Simons term plays the role of a functional magnetic field, i.e. there is a non-trivial curvature in configuration space, representing entanglement between the two sectors of the theory. Exactly as in the standard problem of Landau levels we can define lowering and raising operators

$$\begin{aligned} \mathcal{A}^i &= \sqrt{\frac{\pi}{\bar{q}e_q e_v}} \left( e_v \Pi_a^i - i e_q \epsilon^{ij} \Pi_b^j \right) , \\ \mathcal{A}^{i\dagger} &= \sqrt{\frac{\pi}{\bar{q}e_q e_v}} \left( e_v \Pi_a^i + i e_q \epsilon^{ij} \Pi_b^j \right) , \end{aligned} \quad (52)$$

with commutation relation

$$[\mathcal{A}^i(\mathbf{x}), \mathcal{A}^{j\dagger}(\mathbf{y})] = \delta^{ij} \delta^2(\mathbf{x} - \mathbf{y}) . \quad (53)$$

In terms of these, the Hamiltonian takes the familiar form

$$H = m \sum_i \int d^2 \mathbf{x} \left( \mathcal{A}^{i\dagger}(\mathbf{x}) \mathcal{A}^i(\mathbf{x}) + \bar{q} \delta^{ij} \delta^2(\mathbf{0}) \right) , \quad (54)$$



where the second term represents the infinite ground state energy that has to be subtracted.

Finally, the Gauss law operators, implementing gauge invariance, are the constraints associated with the Lagrange multipliers  $a_0$  and  $b_0$ ,

$$\begin{aligned} G_a &\equiv \partial_i \mathcal{P}_a^i + \frac{\bar{q}}{4\pi} \partial_i \epsilon^{ij} b^j, \\ G_b &\equiv \partial_i \mathcal{P}_b^i + \frac{\bar{q}}{4\pi} \partial_i \epsilon^{ij} a^j. \end{aligned} \quad (55)$$

At the quantum level these constraints must be imposed as conditions on physical states:

$$G_a \Psi[a^i, b^i] = 0, \quad G_b \Psi[a^i, b^i] = 0. \quad (56)$$

The wave functional  $\Psi_0$  of the topological insulator is thus given by the functional first Landau level, defined by  $\mathcal{A}^i(\mathbf{x})\Psi_0[a^i, b^i] = 0$ , subject to the gauge constraints (56).

The crucial point is that, given the impossibility of neglecting regularizing kinetic terms to obtain normalizable quantum states, the Gauss law constraints (55) do not coincide with the physical charge and vortex densities  $j^0$  and  $\phi^0$ , those that condense in the superconductor and superinsulator, respectively. While the two gauge symmetries are linearly realized, the physical charges and vortices (those that eventually condense) do fluctuate in the topological insulator.

Following [3, 9] we write the ground state functional as the product of a cocycle and a part that depends only on the transverse components of the two dynamical variables,  $a_T^i$  and  $b_T^i$ :

$$\begin{aligned} \Psi_0[a^i, b^i] &= \exp i\chi[a^i, b^i] \Phi(a_T^i, b_T^i), \\ \chi[a^i, b^i] &= \frac{\bar{q}}{4\pi} \int d^2\mathbf{x} [b \frac{\partial_i}{\Delta} a^i + a \frac{\partial_i}{\Delta} b^i], \\ \Phi[a_T^i, b_T^i] &= \exp \frac{-\bar{q}}{4\pi} \int d^2\mathbf{x} \left( \frac{e_v}{e_q} (a_T^i)^2 + \frac{e_q}{e_v} (b_T^i)^2 \right), \end{aligned} \quad (57)$$

where  $a = \epsilon^{ij} \partial_i a^j$ ,  $b = \epsilon^{ij} \partial_i b^j$ ,  $\Delta = \partial_i \partial_i$  and  $a_T^i = P^{ij} a^j$ ,  $b_T^i = P^{ij} b^j$ , with the projector  $P^{ij}$  onto the transverse part of the gauge fields given by  $P^{ij} = \left( \delta^{ij} - \frac{\partial^i \partial^j}{\Delta} \right)$ . Using the Hodge decomposition for the spatial components of the two gauge fields  $a^i$  and  $b^i$ :

$$\begin{aligned} a^i &= \partial_i \xi + \epsilon^{ij} \partial_j \phi, \\ b^i &= \partial_i \lambda + \epsilon^{ij} \partial_j \psi, \end{aligned} \quad (58)$$

we can rewrite  $\Psi_0[a^i, b^i]$  as:

$$\begin{aligned} \Psi_0[a^i, b^i] &= \exp \frac{i\bar{q}}{4\pi} \int d^2\mathbf{x} [\psi \Delta \xi + \phi \Delta \lambda] \exp \frac{-\bar{q}}{4\pi} \int d^2\mathbf{x} \left[ g(\partial_i \phi)^2 + \frac{1}{g}(\partial_i \psi)^2 \right] \\ &= \exp \frac{i\sqrt{\bar{q}}}{2} \int d^2\mathbf{x} [\lambda \phi^0 + \xi j^0] \exp \frac{-\bar{q}}{4\pi} \int d^2\mathbf{x} \left[ g(\partial_i \phi)^2 + \frac{1}{g}(\partial_i \psi)^2 \right], \end{aligned} \quad (59)$$

where  $j^0$  and  $\phi^0$  represent the charge- and vortex-number densities, respectively.

When the gauge symmetries are non-compact, the cocycle in (59) never contributes to correlation functions. Things change, however, when the two symmetries are compact  $U(1)$  and topological excitations are present. In this case the fields  $\xi$  and  $\lambda$  are angles and the identity  $\epsilon^{ij} \partial_i \partial_j \theta = 2\pi \delta^2(\mathbf{x})$ , in polar coordinates  $r, \theta$ , implies thus the existence of quantized vortices and point charges. As we now show, in this  $U(1) \times U(1)$  case, the fields  $\xi$  and  $\lambda$  become new dynamical fields describing the dynamics of these additional degrees of freedom.

To derive this, let us consider charge sector observables in an entangled mixed state in which the gauge degrees of freedom are considered as the non-observed environment. We focus thus on wave functionals which are superpositions of different vortex states and of different transverse gauge field  $\phi$  components. The expectation values of charge sector operators  $O(\psi, \lambda)$  in this mixed state are then given by

$$\begin{aligned} \langle O \rangle &= \frac{1}{Z} \int \mathcal{D}\psi \mathcal{D}\lambda \mathcal{D}\phi \sum_N \frac{z^N}{N!} \sum_{\mathbf{x}_1 \dots \mathbf{x}_N} \sum_{\Phi_1 \dots \Phi_N = \pm 1} O(\psi, \lambda) \\ &= e^{\frac{i}{2} \sum_i \int d^2\mathbf{x} \lambda(\mathbf{x}) \Phi(\mathbf{x}, \mathbf{x}_i, \Phi_i) + i \int d^2\mathbf{x} \frac{\bar{q}}{4\pi} \lambda \Delta \phi} e^{\frac{-\bar{q}}{2\pi} \int d^2\mathbf{x} \left( \frac{g}{4} (\partial_i \phi)^2 + \frac{1}{g} (\partial_i \psi)^2 \right)}. \end{aligned} \quad (60)$$

where  $Z$  is the normalization factor,  $\Phi(\mathbf{x}, \mathbf{x}_i, \Phi_i) = \Phi_i \delta^2(\mathbf{x} - \mathbf{x}_i)$  and  $\phi$  denotes the difference  $\phi = \phi_{\text{bra}} - \phi_{\text{ket}}$  between bra and ket states. We have also used the dilute vortex approximation in which only interferences between vortex

states differing by one unit are taken into account. The quantum fugacity parameter  $z$  governs the entanglement. For large  $z$  we have a highly entangled state between vortex and charge degrees of freedom, for  $z \rightarrow 0$ , vortices are liberated as independent degree of freedom. At this stage both the integration over the transverse field  $\phi$  and the summation over vortex interference configurations can be done explicitly [1], with the result

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}\psi \mathcal{D}\lambda \, O(\psi, \lambda) \, e^{-\int d^2\mathbf{x} \, \frac{g}{2\pi} (\partial_i \psi)^2 + \frac{g}{2\pi} (\partial_i \lambda)^2 - 2z \cos \lambda} , \quad (61)$$

Charge observables in the entangled mixed state are thus determined by the classical partition function of the 2D sine-Gordon model, or equivalently the 2D XY model [14]. While the original  $\psi$  field plays the role of the spin waves, the new, dynamical sine-Gordon field  $\lambda$  embeds the dynamics of point charges (topological excitations in this formulation). Of course, the inverse happens in the entangled mixed state in which charges are the fluctuating environment. In this case, it is the field  $\phi$  that plays the role of spin waves and  $\xi$  is the new dynamical field describing vortices.

## BKT TRANSITIONS AND QUANTUM CORRELATION FUNCTIONS

The 2D XY model undergoes the famed BKT transition [12, 13]. In this case, however, we are dealing with a quantum BKT transition, since it is the SIT resistance parameter  $1/g$  that plays the role of effective temperature for the vortex entanglement, superconducting transition, while  $g$  plays the same role for the charge entanglement, superinsulating transition. In the superconducting and superinsulating phases, existing at very high and very low  $g$ , respectively, the charge and vortex correlation functions are algebraic. The bosonic topological insulator exists, instead at intermediate values of  $g \approx 1$  so that both charges and vortices are in their “high-temperature”, free phase with screened correlation functions. In this phase, Bose condensation of either charges and vortices is prevented by strong quantum correlations and we have the screened correlations

$$\begin{aligned} \langle j^0(\mathbf{x}) j^0(\mathbf{y}) \rangle &\propto \exp \left( -\frac{|\mathbf{x} - \mathbf{y}|}{\xi(g)} \right) , \\ \langle \phi^0(\mathbf{x}) \phi^0(\mathbf{y}) \rangle &\propto \exp \left( -\frac{|\mathbf{x} - \mathbf{y}|}{\xi(1/g)} \right) , \end{aligned} \quad (62)$$

where the BKT correlation length  $\xi(g)$  diverges for  $g$  decreasing and approaching the critical value for vortex condensation and  $\xi(1/g)$  increases in the opposite limit. The Bose metal quantum state near the superconducting transition is thus a bosonic topological insulator matrix with superconducting “bubbles” of typical dimension  $\xi(1/g)$ .

## SCALING AT THE QUANTUM BKT TRANSITIONS

In this section we derive the correct scaling relations for quantum BKT transitions. The original Fisher scaling argument [15] is predicated on the fact that, in the vicinity of a quantum critical point  $B_{\text{cr}}$  there is a characteristic length scale  $\xi$  scaling with a critical exponent  $\nu$  as  $\xi \propto |B - B_{\text{cr}}|^{-\nu}$  and a characteristic frequency  $\Omega$  scaling as  $\Omega \propto \xi^{-z}$ , where  $z$  is the dynamical critical exponent. Furthermore, near zero, the temperature  $T$  should scale as the frequency and thus  $T \propto \xi^{-z}$ . Using the simple fact that

$$|B - B_{\text{cr}}| \propto \xi^{\frac{-1}{\nu}} \propto T^{\frac{1}{z\nu}} , \quad (63)$$

we obtain Fisher’s result that the correct scaling variable is  $|B - B_{\text{cr}}|/T^{1/\nu z}$ . This, however, is valid only for second-order transitions characterized by a finite critical exponent  $\nu$ . BKT transitions are of infinite order and, formally, they have critical exponent  $\nu = \infty$ , corresponding to a singular behaviour at criticality. To see what this means, we have to repeat the above analysis with the BKT scaling

$$\xi \propto e^{\sqrt{\frac{B^*}{|B - B_{\text{cr}}|}}} , \quad (64)$$

where  $B^*$  is a constant with dimension of magnetic field. Requiring that temperature scales as the characteristic frequency we obtain

$$T = T_0 \, e^{-z \sqrt{\frac{B^*}{|B - B_{\text{cr}}|}}} , \quad (65)$$

where  $T_0$  is a non-universal temperature. We now follow the same procedure as in the Fisher scaling argument by expressing

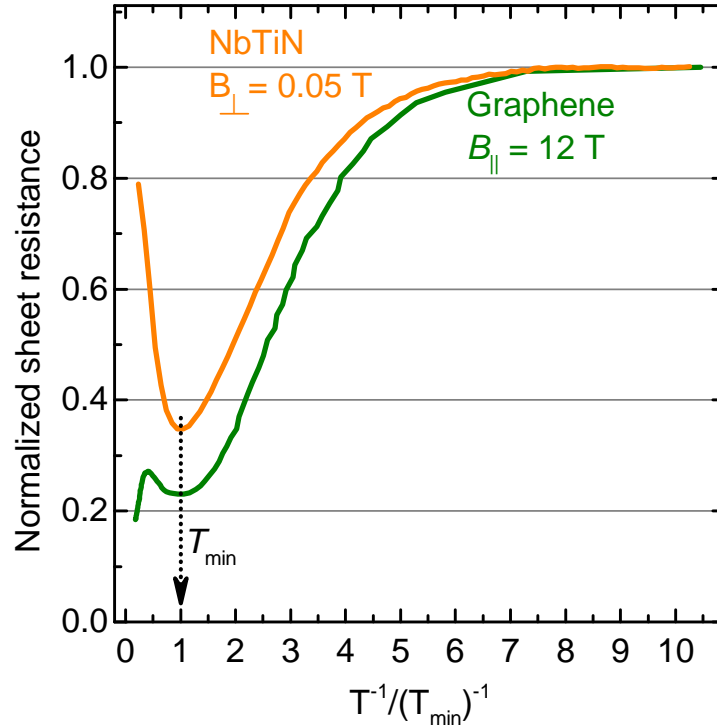
$$\ln\left(\frac{T}{T_0}\right) = -z\sqrt{\frac{B^*}{|B - B_{\text{cr}}|}}, \quad (66)$$

which leads immediately to

$$|B - B_{\text{cr}}| \left( \ln\left(\frac{T}{T_0}\right) \right)^2 = z^2 B^*, \quad (67)$$

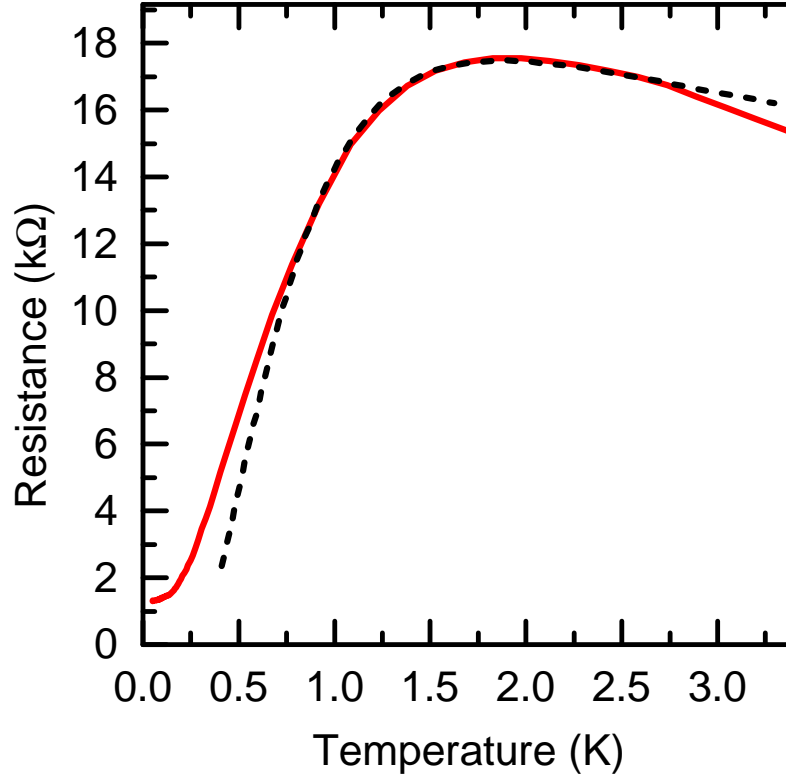
which shows that this quantity,  $z^2 B^*$  is the correct scaling variable in the case of a quantum BKT transition. Unfortunately, as already mentioned, the quantity  $T_0$  is non-universal and must be fitted.

### Low-temperature resistance in NbTiN and double belayer graphene



**Figure S 1: Sheet resistance vs  $1/T$  plots in NbTiN and twisted double belayer graphene (TDBG)** Representative  $R_{\square}$  vs.  $1/T$  plots for fields  $B_{\perp} = 0.05$  T for NbTiN and  $B_{\parallel} = 0.05$  for TDBG (sample No. 2, see main text). The temperature scales are normalized with respect to positions of the minima, resistances are normalized with respect to their saturation values. Importantly, normalized temperatures of the saturation coincide for both systems indicating a universal topological character of the quantum BKT transitions confining the domain of the existence of the bosonic topological insulators.

### Superconducting fit of the resistance curve at zero magnetic field



**Figure S 2: Zero magnetic field sheet resistance.** Zero magnetic field sheet resistance vs.  $T$  from Fig. 3b of the main text replotted in the linear scale (red solid line) for the 10 nm thick NbTiN film. The dashed line shows the standard fit by superconducting fluctuations and quantum corrections [16]. The deviation from the fit is due to incremental contribution from fluctuation vortex motion and saturation at lowest temperature indicates that fluctuation vortex motion is dominated by quantum tunneling of vortices.

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